

A Note on Weight Distributions of Irreducible Cyclic Codes

Baocheng Wang, Chunming Tang, Yanfeng Qi, Yixian Yang, Maozhi Xu

Abstract—Usually, it is difficult to determine the weight distribution of an irreducible cyclic code. In this paper, we discuss the case when an irreducible cyclic code has the maximal number of distinct nonzero weights and give a necessary and sufficient condition. In this case, we also obtain a divisible property for the weight of a codeword. Further, we present a necessary and sufficient condition for an irreducible cyclic code with only one nonzero weight. Finally, we determine the weight distribution of an irreducible cyclic code for some cases.

Index Terms—Cyclotomic fields, irreducible cyclic codes, Gauss sums, Gaussian periods, weight distributions.

I. INTRODUCTION

Let p be a prime, $q = p^s$ and $r = q^m$, $k = sm$, where s and m are two positive integers. Suppose that an integer N satisfies $N|(r-1)$. Let $n = \frac{r-1}{N}$ and $N_2 = (N, \frac{r-1}{q-1})$. Given two integers a and b , which are coprime, ord_{ab} denotes the order of a module b . Consider the finite field $GF(r)^* = \langle \alpha \rangle$. Let $\theta = \alpha^N$. The set

$$\mathcal{C}(r, N) = \{(Tr_{r/q}(\beta), Tr_{r/q}(\beta\theta), \dots, Tr_{r/q}(\beta\theta^{n-1})) : \beta \in GF(r)\}$$

is called an irreducible cyclic $[n, m_0]$ code, where $Tr_{r/q}$ is the trace function from $GF(r)$ onto $GF(q)$ and $m_0 = \dim_{GF(q)}(GF(q)(\theta))$ [6]. Then $m_0|m$. Generally, $m_0 = m$ is satisfied. Hence, we just consider the case $m_0 = m$ in this paper.

The weight distribution of $\mathcal{C}(r, N)$ attracts much interest. Generally, it is difficult to determine the weight distribution [14]. For any $\beta \in GF(r)^*$, the Hamming weight of any codeword

$$c(\beta) = (Tr_{r/q}(\beta), Tr_{r/q}(\beta\theta), \dots, Tr_{r/q}(\beta\theta^{n-1}))$$

in the code $\mathcal{C}(r, N)$ can be represented by a linear combination of Gauss sums via Fourier transform [15], [16], [17]. Hence, Gauss sums can be used to analyze the weight distribution of the irreducible cyclic codes. Some main results on the weight distribution are list here.

- Let $N|(q^j + 1)$, where $j|\frac{m}{2}$. This is called the semi-primitive case [2], [4], [15]. Then $\mathcal{C}(r, N)$ is a two weight code.

- Let N be a prime satisfying $N \equiv 3 \pmod{4}$ and $\text{ord}_N(q) = (N-1)/2$. Baumert and Mykkeltveit [3] determined the corresponding weight distribution.

B. Wang and Y. Yang are with Department of Computer, Beijing University of Posts and Telecommunications, Beijing, 100876, China

C. Tang, Y. Qi and M. Xu is with the School of Mathematical Sciences, Peking University, 100871, China

C. Tang's e-mail: tangchunmingmath@163.com

- Let $N = 2$. Baumert and McEliece [2] gave the corresponding weight distribution.

- Let $N = 3, 4$, Ding [5] determined the weight distribution.

- Let $N_2 = 1, 2$, Ding [5] determined the weight distribution.

Schmidt and White [20] got necessary and sufficient conditions for an irreducible cyclic code having at most two weights. This paper considers two other cases. One is that a irreducible cyclic code has the maximal number of distinct nonzero weights; the other is that a irreducible cyclic code has only one nonzero weight. More results on the weight distribution can be found in [1], [8], [12], [18], [20], [22].

In this paper, we simplify the formula of $wt(c(\beta))$ [5], which is represented by Gauss periods. From the simplified formula, we find that determining the weight distribution of $\mathcal{C}(r, N)$ is equivalent to determining the weight distribution of $\mathcal{C}(r, N_2)$. Further, determining the weight distribution of $\mathcal{C}(r, N)$ is reduced to the factorization of $\psi_{(N_2, r)}^*(X)$. From cyclotomic fields and Stickelberger theorem on Gauss sums, We present a necessary and sufficient condition for an irreducible cyclic code having the maximal number of different nonzero weights and obtain a divisible property of the weight of a codeword. Then we generalize the results of Ding [5] to cases $N_2 = 3, 4$ and give the necessary and sufficient conditions for a code with only one nonzero weight. Finally, we present the weight distributions of the irreducible cyclic codes for cases $N = 5, 6, 8, 12$.

II. PRELIMINARIES

For determining the weight distribution, we first recall cyclotomic classes, Gauss sums and reduced period polynomials.

The definition of cyclotomic classes of order N in $GF(r)$ is the coset

$$C_i^{(N, r)} = \alpha^i \langle \alpha^N \rangle.$$

When $i \equiv j \pmod{N}$, $C_i^{(N, r)} = C_j^{(N, r)}$. If $\beta \in C_i^{(N, r)}$, we let $i(\beta) = i$.

The Gauss periods are

$$\eta_i^{(N, r)} = \sum_{x \in C_i^{(N, r)}} \mu(x),$$

where $\mu(x) = \zeta_p^{Tr_{r/p}(x)}$, $\zeta_p = \exp(2\pi i/p)$ and $i = \sqrt{-1}$.

Let S be a subset of $GF(r)$, $\mu(S)$ denotes $\sum_{x \in S} \mu(x)$. Then $\eta_i^{(N, r)} = \mu(C_i^{(N, r)})$. Let $\eta_i^{*(N, r)} = 1 + N\eta_i^{(N, r)}$, then $\eta_i^{*(N, r)}$ is called the reduced period. The reduced periods polynomial is of the form

$$\psi_{(N, r)}^*(X) = (X - \eta_0^{*(N, r)}) \cdots (X - \eta_{N-1}^{*(N, r)}).$$

A. Gauss sums

In this subsection, we introduce some knowledge on Gauss sums [13].

Let $\chi : GF(r)^* \rightarrow C^*$ be a character of $GF(r)^*$, where C is the complex field. Then a Gauss sum is defined by

$$g(\chi) = - \sum_{\beta \in GF(r)} \chi(\beta) \mu(\beta).$$

For any $c_2(\beta) \in \mathcal{C}(r, N_2)$, we have the following McEliece's identity [16]

$$wt(c_2(\beta)) = \frac{q-1}{N_2 q} \left(r + \sum_{\chi^{N_2=1}, \chi \neq 1} g(\chi) \overline{\chi}(\beta) \right). \quad (1)$$

To determine $wt(c_2(\beta))$, we can utilize properties of the Gauss sum $g(\chi)$.

Given an integer t , let $\zeta_t = \exp(2\pi i/t)$. Let \wp be a prime ideal of $Q(\zeta_{r-1})$ over p and $\tilde{\wp}$ is a prime ideal of $Q(\zeta_{r-1}, \zeta_p)$ over \wp . An integer h ($0 \leq h < r-1$) can be represented by $h = h_0 + h_1 p + \dots + h_{k-1} p^{k-1}$. Then we let $s(h) = \sum_{i=0}^{k-1} h_i p^i$.

There is an isomorphism

$$Z[\zeta_{r-1}]/\wp \longrightarrow GF(r).$$

Note that all the $r-1$ -th root of unity are different module \wp . Then we have the following isomorphism [23]

$$\omega(\cdot) : GF(r)^* \longrightarrow \{\zeta_{r-1}^0, \zeta_{r-1}^1, \dots, \zeta_{r-1}^{r-2}\},$$

which satisfies

$$\omega(\beta) \pmod{\wp} = \beta \in GF(r)^*.$$

From Stickelberger's theorem [21], we have

$$v_{\tilde{\wp}}(g(\omega^{-h})) = s(h). \quad (2)$$

Further, we have the following relation from Lang [10], [11]

$$g(\omega^{-h}) \equiv \frac{(\zeta_p - 1)^{s(h)}}{(h_0!) \dots (h_{k-1}!)} \pmod{\tilde{\wp}^{s(h)+1}}. \quad (3)$$

Some results on the factorization of some reduced period polynomials are listed here.

B. The factorization of $\psi_{(N,r)}^*$ for cases $N = 2, 3, 4$ [19]

Lemma 2.1: Let $2 \mid \frac{r-1}{p-1}$. Then we have the following results on the factorization of $\psi_{(2,r)}^*(X)$

$$\psi_{(2,r)}^*(X) = (X + \sqrt{r})(X - \sqrt{r}).$$

Lemma 2.2: Let $3 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(3,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{3}$, then $3 \mid k$, and

$$\begin{aligned} \psi_{(3,r)}^*(X) &= (X - cr^{1/3})(X + \frac{1}{2}(c + 9d)r^{1/3}) \\ &\quad \times (X + \frac{1}{2}(c - 9d)r^{1/3}) \end{aligned}$$

where c and d are given by $4p^{k/3} = c^2 + 27d^2$, $c \equiv 1 \pmod{3}$, and $\gcd(c, p) = 1$.

(b) If $p \equiv 2 \pmod{3}$, then $2 \mid k$, and

$$\psi_{(3,r)}^*(X) = (X + (-1)^{k/2} \sqrt{r})(X - (-1)^{k/2} \sqrt{r})^2.$$

Lemma 2.3: Let $4 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(4,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{4}$, then $4 \mid k$, and

$$\begin{aligned} \psi_{(4,r)}^*(X) &= (X + \sqrt{r} + 2r^{1/4}u)(X + \sqrt{r} - 2r^{1/4}u) \times \\ &\quad (X - \sqrt{r} + 4r^{1/4}v)(X - \sqrt{r} - 4r^{1/4}v) \end{aligned}$$

where u and v are given by $p^{k/2} = u^2 + 4v^2$, $u \equiv 1 \pmod{4}$ and $\gcd(u, p) = 1$.

(b) If $p \equiv 3 \pmod{4}$, then $2 \mid k$, and

$$\psi_{(4,r)}^*(X) = (X + (-1)^{k/2} 3\sqrt{r})(X - (-1)^{k/2} \sqrt{r})^3.$$

C. The factorization of $\psi_{(5,r)}^*$

In order to describe the explicit factorization of the quintic period polynomials for finite fields, we require to discuss integer solutions of Dickson's system.

$$\begin{cases} 16 \sqrt[5]{r} = x^2 + 125w + 50v^2 + 50u^2, \\ xw = v^2 - 4vu - u^2, \\ x \equiv -1 \pmod{5}. \end{cases}$$

Let σ be a non-singular linear transformation of order 4.

$$\sigma(x, w, v, u) = (x, -w, -u, v).$$

Then we have the following lemma for the explicit factorization of period polynomials [9].

Lemma 2.4: Let $5 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(5,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{5}$, then $5 \mid k$ and

$$\psi_{(5,r)}^*(X) = (X + \frac{\sqrt[5]{r}}{16}(x^3 - 25L)) \prod_{i=0}^3 (X - \frac{\sqrt[5]{r}}{64} \sigma^i(X^3 - 25M)),$$

where (x, w, v, u) is the integer solution of Dickson's system satisfying the condition $p \nmid (x^2 - 125w^2)$ and

$$\begin{aligned} L &= 2x(v^2 + u^2) + 5w(11v^2 - 4vu - 11u^2), \\ M &= 2x^2u + 7xv^2 + 20xvu - 3xu^2 + 125w^3 \\ &\quad + 200w^2v - 150w^2u + 5wv^2 - 20wvu \\ &\quad - 105wu^2 - 40v^3 - 60v^2u + 120vu^2 + 20u^3. \end{aligned}$$

(b) If $p \not\equiv 1 \pmod{5}$, then

$$\psi_{(5,r)}^*(X) = \begin{cases} (X - r^{1/2})^4(X + 4r^{1/2}), & \text{if } k/\text{ord}_5 p \text{ is even} \\ (X + r^{1/2})^4(X - 4r^{1/2}), & \text{if } k/\text{ord}_5 p \text{ is odd} \end{cases}$$

D. The factorization of $\psi_{(N,r)}^*$ for cases $N = 6, 8, 12$

To determine the factorization of the reduced period polynomial $\psi_{(N,r)}^*$ for cases $N = 6, 8, 12$ [7], we need to introduce some notations.

Let $G := \alpha^{\frac{r-1}{q-1}}$ and Z be the least positive integer satisfying $G^Z = 2$.

When $p \equiv 1 \pmod{3}$. Then there exist two unique integers r_3 and s_3 satisfying

$$\begin{aligned} 4p &= r_3^2 + 3s_3^2 \\ r_3 &\equiv 1 \pmod{3} \\ s_3 &\equiv 0 \pmod{3} \\ 3s_3 &\equiv (2G^{(p-1)/3} + 1)r_3 \pmod{p} \end{aligned}$$

Let $\lambda = (r_3 + i\sqrt{3}s_3)/2$ and a sequence related to λ be

$$V_{j,n} = \zeta_6^{-j} \lambda^n + \zeta_6^j \bar{\lambda}^n.$$

When the prime $p \equiv 1 \pmod{4}$, there exist two unique integers a_4 and b_4 satisfying

$$\begin{aligned} p &= a_4^2 + b_4^2 \\ a_4 &\equiv -(-1)^Z \pmod{4} \\ b_4 &\equiv a_4 G^{(p-1)/4} \pmod{p}. \end{aligned}$$

Then we can define $\pi = a_4 + ib_4$ and sequences related to π .

$$\begin{aligned} Q_n &= \pi^n + \bar{\pi}^n, & P_n &= -i(\pi^n - \bar{\pi}^n), \\ Q_{j,n} &= \zeta_4^{-j} \pi^n + \zeta_4^j \bar{\pi}^n, & P_{j,n} &= -i(\zeta_4^{-j} \pi^n - \zeta_4^j \bar{\pi}^n). \end{aligned}$$

When $p \equiv 1 \pmod{8}$, there exist two unique integers a_8 and b_8 satisfying

$$\begin{aligned} p &= a_8^2 + 2b_8^2 \\ a_8 &\equiv -1 \pmod{4} \\ 2b_8 &\equiv (G^{(p-1)/8} + G^{3(p-1)/8})a_8 \pmod{p}. \end{aligned}$$

When $p \equiv 3 \pmod{8}$, there exist two unique integers a_8 and b_8 satisfying

$$\begin{aligned} p &= a_8^2 + 2b_8^2 \\ a_8 &\equiv (-1)^{(p-3)/8} \pmod{4} \\ 2b_8 &\equiv (\alpha^{(q-1)/8} - \alpha^{(1-q)/8})a_8 \pmod{p}. \end{aligned}$$

For the two cases, we define $\sigma = a_8 + ib_8\sqrt{2}$ and sequences related to σ .

$$T_n = \sigma^n + \bar{\sigma}^n, \quad S_n = (\sigma^n - \bar{\sigma}^n)/(i\sqrt{2}).$$

Lemma 2.5: Let $6 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(6,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{6}$, then $6 \mid k$ and

$$\psi_{(6,r)}^*(X) = (X - \eta_1^{*(6,r)}) \cdots (X - \eta_6^{*(6,r)}),$$

where $\eta_j^{*(6,r)} = -(-1)^{tk/2} p^{k/6} V_{j,2k/3} - p^{k/3} V_{2j,k/3} - (-1)^{j+tk/2} p^{k/2}$, $t = \frac{p-1}{6}$ and $V_{j,n}$ are defined above.

(b) If $p \equiv 5 \pmod{6}$, then

$$\psi_{(6,r)}^*(X) = (X - (-1)^{k/2} p^{k/2})^5 (X + 5(-1)^{k/2} p^{k/2}).$$

Lemma 2.6: Let $8 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(8,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{8}$, then $8 \mid k$ and

$$\psi_{(8,r)}^*(X) = (X - \eta_1^{*(8,r)}) \cdots (X - \eta_8^{*(8,r)}),$$

where $\eta_j^{*(8,r)} = -p^{k/2} - p^{k/4} Q_{j,k/2} - p^{k/8} AB$ and A, B are defined as follows: If j is even, $A = Q_{j/2,k/4}$ and $B = T_{k/2}$. If j is odd, $A = (-1)^{[j/4]} Q_{0,k/4} + (-1)^{[(j-2)/4]} P_{0,k/4}$ and $B = S_{k/2}$.

(b) If $p \equiv 3 \pmod{8}$, then $4 \mid k$ and

$$\psi_{(8,r)}^*(X) = (X - \xi_1)^2 (X - \xi_2)^2 (X - \xi_3)^2 (X - \xi_4) (X - \xi_5),$$

where $\xi_1 = -2p^{k/2} S_{k/2} + p^{k/2}$, $\xi_2 = 2p^{k/2} S_{k/2} + p^{k/2}$, $\xi_3 = p^{k/2}$, $\xi_4 = 2p^{k/4} T_{k/2} - 3p^{k/2}$ and $\xi_5 = -2p^{k/4} T_{k/2} - 3p^{k/2}$.

(c) If $p \equiv 5 \pmod{8}$, then $8 \mid k$ and

$$\begin{aligned} \psi_{(8,r)}^*(X) &= (X - \xi_1)^2 (X - \xi_2)^2 (X - \xi_3) (X - \xi_4) \\ &\quad \times (X - \xi_5) (X - \xi_6), \end{aligned}$$

where $\xi_1 = p^{k/2} - p^{k/4} P_{k/2}$, $\xi_2 = p^{k/2} + p^{k/4} P_{k/2}$, $\xi_3 = -p^{k/2} - p^{k/4} Q_{k/2} - 2p^{3k/8} Q_{k/4}$, $\xi_4 = -p^{k/2} - p^{k/4} Q_{k/2} + 2p^{3k/8} Q_{k/4}$, $\xi_5 = -p^{k/2} + p^{k/4} Q_{k/2} - 2p^{3k/8} P_{k/4}$ and $\xi_6 = -p^{k/2} + p^{k/4} Q_{k/2} + 2p^{3k/8} P_{k/4}$.

(d) If $p \equiv 7 \pmod{8}$, then $2 \mid k$ and

$$\psi_{(8,r)}^*(X) = (X - (-1)^{k/2} p^{k/2})^7 (X + 7(-1)^{k/2} p^{k/2}).$$

Lemma 2.7: Let $12 \mid \frac{r-1}{p-1}$. We have the following results on the factorization of $\psi_{(12,r)}^*(X)$.

(a) If $p \equiv 1 \pmod{12}$, then $12 \mid k$ and

$$\psi_{(12,r)}^*(X) = (X - \eta_1^{*(12,r)}) \cdots (X - \eta_{12}^{*(12,r)}),$$

where $\eta_j^{*(12,r)} = -p^{k/12} Q_{j,k/2} V_{-j,k/3} - p^{k/4} Q_{j,k/2} - p^{k/6} V_{j,2k/3} - p^{k/3} V_{2j,k/3} - (-1)^j p^{k/2}$.

(b) If $p \equiv 5 \pmod{12}$, then $4 \mid k$ and

$$\begin{aligned} \psi_{(12,r)}^*(X) &= (X - \xi_1)^2 (X - \xi_2)^2 (X - \xi_3)^2 (X - \xi_4)^2 \\ &\quad \times (X - \xi_5) \cdots (X - \xi_8), \end{aligned}$$

where $\xi_1 = Q_{k/2} p^{k/4} ((-1)^{k/4} - 1) + p^{k/2}$, $\xi_2 = -Q_{k/2} p^{k/4} ((-1)^{k/4} - 1) + p^{k/2}$, $\xi_3 = P_{k/2} p^{k/4} ((-1)^{k/4} + 1) + p^{k/2}$, $\xi_4 = -P_{k/2} p^{k/4} ((-1)^{k/4} + 1) + p^{k/2}$, $\xi_5 = P_{k/2} p^{k/4} (2(-1)^{k/4} - 1) + p^{k/2}$, $\xi_6 = -P_{k/2} p^{k/4} (2(-1)^{k/4} - 1) + p^{k/2}$, $\xi_7 = Q_{k/2} p^{k/4} (2(-1)^{k/4} + 1) - 5p^{k/2}$ and $\xi_8 = -Q_{k/2} p^{k/4} (2(-1)^{k/4} - 1) - 5p^{k/2}$.

(c) If $p \equiv 7 \pmod{12}$. Let $\rho = 2(-1)^{k(p+5)/6}$, then $6 \mid k$ and

$$\psi_{(12,r)}^*(X) = (X - \eta_1^{*(12,r)}) \cdots (X - \eta_j^{*(12,r)}),$$

where $\eta_j^{*(12,r)}$ is defined as follows: If j is odd, $\eta_j^{*(12,r)} = -(-1)^{k/2} p^{k/6} V_{j,2k/3} - p^{k/3} V_{2j,k/3} + (-1)^{k/2} p^{k/2}$; if $2 \parallel j$, $\eta_j^{*(12,r)} = -(-1)^{k/2} p^{k/6} V_{j,2k/3} + p^{k/3} V_{2j,k/3} (\rho - 1) + p^{k/2} (\rho - (-1)^{k/2})$; if $4 \mid j$, $\eta_j^{*(12,r)} = -(-1)^{k/2} p^{k/6} V_{j,2k/3} - p^{k/3} (\rho + 1) V_{2j,k/3} - p^{k/2} (\rho + (-1)^{k/2})$.

(d) If $p \equiv 11 \pmod{12}$, then $2 \mid k$ and

$$\psi_{(12,r)}^*(X) = (X - (-1)^{k/2} p^{k/2})^{11} (X + 11(-1)^{k/2} p^{k/2}).$$

III. SOME GENERAL RESULTS ON THE WEIGHT DISTRIBUTION

To discuss the weight distribution, we first consider

$$Z(r, \beta) = \#\{Tr_{r/q}(\beta x^N) = 0 : x \in GF(r)\}.$$

Then $wt(c(\beta)) = n - \frac{Z(r, \beta) - 1}{N}$.

For $Z(r, \beta)$, we have the following formula [5].

$$\begin{aligned} Z(r, \beta) &= \frac{1}{q} \sum_{y \in GF(q)} \sum_{x \in GF(r)} \zeta_p^{Tr_{q/p}(y Tr_{r/q}(\beta x^N))} \\ &= \frac{1}{q} [q + r - 1 + N \sum_{y \in GF(q)^*} \sum_{x \in C_0^{N,r}} \mu(y\beta x)]. \end{aligned}$$

For the simplification of this formula, we introduce some lemmas.

Lemma 3.1: We use the assumptions and notations above, then

- (a) $\#(GF(q)^* \cap C_0^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}$.
- (b) $GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$ if and only if $(N, \frac{r-1}{q-1}) | i$.
- (c) If $GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$, then $\#(GF(q)^* \cap C_i^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}$.

Proof: (a) Since $GF(q)^*$ and $C_0^{(N,r)}$ are subgroups of the cyclic group $GF(r)^*$, where

$$GF(q)^* = \langle \alpha^{\frac{r-1}{q-1}} \rangle \quad \text{and} \quad C_0^{(N,r)} = \langle \alpha^N \rangle,$$

then

$$GF(q)^* \cap C_0^{(N,r)} = \langle \alpha^{[\frac{r-1}{q-1}, N]} \rangle.$$

that is

$$\#(GF(q)^* \cap C_0^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}.$$

(b) If $(N, \frac{r-1}{q-1}) | i$, then there exists two integers a and b satisfying

$$Na + \frac{r-1}{q-1}b = i.$$

Hence

$$\alpha^{\frac{r-1}{q-1}b} = \alpha^{-Na} \cdot \alpha^i \in (GF(q)^* \cap C_i^{(N,r)}).$$

that is

$$GF(q)^* \cap C_i^{(N,r)} \neq \emptyset.$$

If $GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$, there exists two integers a and b satisfying

$$\alpha^{\frac{r-1}{q-1}a} = \alpha^{Nb} \cdot \alpha^i.$$

that is,

$$\alpha^i = \alpha^{\frac{r-1}{q-1}a} \cdot \alpha^{-Nb} \in \langle \alpha^{\frac{r-1}{q-1}}, \alpha^N \rangle.$$

Note that

$$\langle \alpha^{\frac{r-1}{q-1}}, \alpha^N \rangle = \langle \alpha^{(\frac{r-1}{q-1}, N)} \rangle.$$

Hence

$$\alpha^i \in \langle \alpha^{(\frac{r-1}{q-1}, N)} \rangle.$$

Then we have

$$(\frac{r-1}{q-1}, N) | i.$$

(c) If $GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$, we can let $GF(q)^* \cap C_i^{(N,r)} = n_i$.

We first prove that $n_i \geq n_0$. All the elements in $GF(q)^* \cap C_0^{(N,r)}$ are denoted by

$$x_1 = y_1,$$

$$\vdots$$

$$x_{n_0} = y_{n_0},$$

where $x_1, \dots, x_{n_0} \in GF(q)^*$ and $y_1, \dots, y_{n_0} \in C_0^{(N,r)}$. Take an element $z_1 = w_1$ in $GF(q)^* \cap C_i^{(N,r)}$, where $z_1 \in GF(q)^*$

and $w_1 \in C_i^{(N,r)}$. Then $z_1 x_j \in GF(q)^*$ and $w_1 y_j \in C_i^{(N,r)}$. Hence

$$z_1 x_j = w_1 y_j \in GF(q)^* \cap C_i^{(N,r)} \quad (1 \leq j \leq n_0).$$

Thus, we have $n_i \geq n_0$.

We now prove $n_0 \geq n_i$. All the elements in $GF(q)^* \cap C_i^{(N,r)}$ are denoted by

$$z_1 = w_1,$$

$$\vdots$$

$$z_{n_i} = w_{n_i},$$

where $z_1, \dots, z_{n_i} \in GF(q)^*$ and $w_1, \dots, w_{n_i} \in C_i^{(N,r)}$. Then $z_j z_1^{-1} \in GF(q)^*$ and $w_j w_1^{-1} \in C_0^{(N,r)}$. Hence

$$z_j z_1^{-1} = w_j w_1^{-1} \in GF(q)^* \cap C_0^{(N,r)} \quad (1 \leq j \leq n_i).$$

Thus, we have $n_0 \geq n_i$.

Then we have $n_i = n_0$, that is, $\#(GF(q)^* \cap C_i^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}$. ■

Lemma 3.2: Let $\beta \in GF(r)^*$, then

(a) $\beta GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$ if and only if $i \equiv i(\beta) \pmod{N_2}$.

(b) If $\beta GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$, $\#(\beta GF(q)^* \cap C_i^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}$.

Proof: (a) $\beta GF(q)^* \cap C_i^{(N,r)} \neq \emptyset$ if and only if

$$GF(q)^* \cap \beta^{-1} C_i^{(N,r)} = GF(q)^* \cap C_{i-i(\beta)}^{(N,r)} \neq \emptyset.$$

From Lemma 3.1, $GF(q)^* \cap C_{i-i(\beta)}^{(N,r)} \neq \emptyset$ holds if and only if

$$N_2 | (i - i(\beta)).$$

that is

$$i \equiv i(\beta) \pmod{N_2}.$$

Then (a) is proved.

(b) From

$$\#(\beta GF(q)^* \cap C_i^{(N,r)}) = \#(GF(q)^* \cap C_{i-i(\beta)}^{(N,r)})$$

and Lemma 3.1, we can have

$$\#(\beta GF(q)^* \cap C_i^{(N,r)}) = \frac{r-1}{[N, \frac{r-1}{q-1}]}.$$

Lemma 3.3: Let N' be a factor of N , then

$$\eta_i^{(N,r)} + \eta_{i+N'}^{(N,r)} + \dots + \eta_{i+N'(\frac{N}{N'}-1)}^{(N,r)} = \eta_i^{(N',r)}.$$

Proof: This lemma can be proved from the definition of $\eta_i^{(N,r)}$. ■

Theorem 3.4: $\forall \beta \in GF(r)^*$, the Hamming weight of any codeword

$$c(\beta) = (Tr_{r/q}(\beta), Tr_{r/q}(\beta\theta), \dots, Tr_{r/q}(\beta\theta^{n-1}))$$

in the code $\mathcal{C}(r, N)$ is

$$wt(c(\beta)) = \frac{(q-1)(r-1-N_2\eta_{i(\beta)}^{(N_2,r)})}{qN} = \frac{(q-1)(r-\eta_{i(\beta)}^{*(N_2,r)})}{qN}.$$

Proof: The Hamming weight of $c(\beta)$ is

$$wt(c(\beta)) = n - \frac{Z(r, \beta) - 1}{N},$$

where

$$\begin{aligned} Z(r, \beta) &= \frac{1}{q} \sum_{y \in GF(q)} \sum_{x \in GF(r)} \zeta_p^{Tr_{q/p}(y Tr_{r/q}(\beta x^N))} \\ &= \frac{1}{q} [q + r - 1 + N \sum_{y \in GF(q)^*} \sum_{x \in C_0^{N,r}} \mu(y\beta x)]. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{y \in GF(q)^*} \sum_{x \in C_0^{N,r}} \mu(y\beta x) &= \sum_{y \in GF(q)^*} \mu(y\beta C_0^{N,r}) \\ &= a_0(\beta)\eta_0^{(N,r)} + a_1(\beta)\eta_1^{(N,r)} + \cdots + a_{N-1}(\beta)\eta_{N-1}^{(N,r)}, \end{aligned}$$

where $a_i(\beta) = \#(\beta GF(q)^* \cap C_i^{(N,r)})$.

From Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} \sum_{y \in GF(q)^*} \sum_{x \in C_0^{N,r}} \mu(y\beta x) &= \frac{r-1}{[N, \frac{r-1}{q-1}]} [\eta_{i(\beta)}^{(N,r)} + \eta_{i(\beta)+N_2}^{(N,r)} + \cdots + \eta_{i(\beta)+N_2(\frac{N}{N_2}-1)}^{(N,r)}] \\ &= \frac{r-1}{[N, \frac{r-1}{q-1}]} \eta_{i(\beta)}^{(N_2,r)}. \end{aligned}$$

Then

$$\begin{aligned} Z(r, \beta) &= \frac{1}{q} [q + r - 1 + \frac{N(r-1)}{[N, \frac{r-1}{q-1}]} \eta_{i(\beta)}^{(N_2,r)}]. \\ wt(c(\beta)) &= \frac{r-1-Z(r, \beta)+1}{N} \\ &= \frac{r - \frac{1}{q} [q + r - 1 + \frac{N(r-1)}{[N, \frac{r-1}{q-1}]} \eta_{i(\beta)}^{(N_2,r)}]}{N} \\ &= \frac{qr - q - r + 1 - \frac{N(r-1)}{[N, \frac{r-1}{q-1}]} \eta_{i(\beta)}^{(N_2,r)}}{qN} \\ &= \frac{(q-1)(r-1) - (q-1) \frac{N(r-1)}{[N, \frac{r-1}{q-1}]} \eta_{i(\beta)}^{(N_2,r)}}{qN} \\ &= \frac{q-1}{q} \frac{r-1 - N_2 \eta_{i(\beta)}^{(N_2,r)}}{N} \\ &= \frac{q-1}{q} \frac{r - \eta_{i(\beta)}^{*(N_2,r)}}{N}. \end{aligned}$$

Theorem 3.5: Let the factorization of the reduced period polynomial $\psi_{(N_2,r)}^*$ be

$$\psi_{(N_2,r)}^* = (X - \xi_1)^{e_1} \cdots (X - \xi_t)^{e_t},$$

where $e_i \geq 1$ and $e_1 + \cdots + e_t = N_2$, then $\mathcal{C}(r, N)$ is a $[(r-1)/N, m]$ code and its weight distribution is

$$1 + \frac{e_1(r-1)}{N_2} x^{(q-1)(r-\xi_1)/Nq} + \cdots + \frac{e_t(r-1)}{N_2} x^{(q-1)(r-\xi_t)/Nq}.$$

Proof: From Theorem 3.4, this theorem follows directly. ■

Sometimes we also use the weight list

$$wl = [0, (q-1)(r-\xi_1)/Nq, \cdots, (q-1)(r-\xi_t)/Nq]$$

and its corresponding weight distribution frequency

$$Freq(0) = 1, Freq(wl[i+1]) = \frac{e_i(r-1)}{N_2}, i = 1, \cdots, t.$$

to describe the weight distribution.

Corollary 3.6: $\eta_i^{(N_2,r)}$ and $\eta_i^{*(N_2,r)}$ are integers. Further, $\psi_{(N_2,r)}^*$ can be completely factorized over the rational field.

Proof: From the definition of $\eta_i^{(N_2,r)}$ and $\eta_i^{*(N_2,r)}$, they are algebraic integers. Then from Theorem 3.4, they are rational. Hence, they are integers and $\psi_{(N_2,r)}^*$ can be completely factorized over the rational field. ■

Theorem 3.7: $\mathcal{C}(r, N)$ is a code with only one nonzero weight if and only if $N_2 = 1$.

Proof: When $N_2 = 1$, $\mathcal{C}(r, N)$ is obviously a code with only one nonzero weight.

We now prove that $\mathcal{C}(r, N)$ has at least two nonzero weights if $N_2 > 1$. Then we just need to prove $\eta_i^{(N_2,r)}$ are not equal.

If $\eta_1^{(N_2,r)} = \cdots = \eta_{N_2}^{(N_2,r)}$, then we have

$$-1 = \eta_1^{(N_2,r)} + \cdots + \eta_{N_2}^{(N_2,r)} = N_2 \eta_1^{(N_2,r)},$$

that is, $\eta_1^{(N_2,r)} = -\frac{1}{N_2}$. This contradicts that $\eta_1^{(N_2,r)}$ is an integer. Then this theorem follows. ■

From Theorem 3.5, $\mathcal{C}(r, N)$ have at most N_2 distinct nonzero weights. The following theorem gives a necessary and sufficient condition.

Theorem 3.8: The number of distinct nonzero weights in $\mathcal{C}(r, N)$ achieves the maximum N_2 if and only if $p \equiv 1 \pmod{N_2}$.

Proof: Suppose $\mathcal{C}(r, N)$ has N_2 distinct nonzero weights. When $i \not\equiv j \pmod{N_2}$, then

$$\eta_i^{(N_2,r)} \neq \eta_j^{(N_2,r)}.$$

Note that $(C_0^{(N_2,r)})^p = C_0^{(N_2,r)}$. Then

$$C_i^{(N_2,r)} = C_{pi}^{(N_2,r)}.$$

Hence, $i \equiv pi \pmod{N_2}$, that is,

$$(p-1)i \equiv 0 \pmod{N_2}.$$

For any i , $(p-1)i \equiv 0 \pmod{N_2}$. Then we have $p \equiv 1 \pmod{N_2}$.

We now prove that if $p \equiv 1 \pmod{N_2}$, $\mathcal{C}(r, N)$ has the maximal number N_2 of distinct nonzero weights. Then we require to prove $c_2(\zeta_{r-1}^0 \bmod \wp), c_2(\zeta_{r-1}^1 \bmod \wp), \cdots, c_2(\zeta_{r-1}^{N_2-1} \bmod \wp)$ are all distinct.

From the definition of ω , the character χ satisfying $\chi^{N_2} = 1$ can be represented by

$$\chi = \left(\frac{1}{\omega}\right)^{ni}, i = 0, 1, \cdots, N_2 - 1,$$

where $n = \frac{p-1}{N_2} p^0 + \frac{p-1}{N_2} p + \cdots + \frac{p-1}{N_2} p^{k-1}$. Then $s(ni) = \frac{k(p-1)i}{N_2}$. Hence, from (3)

$$g(\omega^{-ni}) \equiv \frac{(\zeta_p - 1)^{\frac{k(p-1)i}{N_2}}}{((\frac{p-1}{N_2}i)!)^k} \bmod \wp^{\frac{k(p-1)i}{N_2} + 1}.$$

Then if $i = 1$, $g(\omega^{-ni}) \pmod{\tilde{\wp}^{\frac{k(p-1)}{N_2}+1}} \equiv \frac{(\zeta_p-1)^{\frac{k(p-1)}{N_2}}}{((\frac{p-1}{N_2})!)^k}$; if $i = 2, \dots, N_2 - 1$, $g(\omega^{-ni}) \pmod{\tilde{\wp}^{\frac{k(p-1)}{N_2}+1}} \equiv 0$.

As a result,

$$\begin{aligned} \sum_{\chi^{N_2=1}, \chi \neq 1} g(\chi) \tilde{\chi}(\zeta_{r-1}^j \pmod{\wp}) &= \sum_{i=1}^{N_2-1} g(\omega^{-ni}) \zeta_{r-1}^{nij} \\ &\equiv \frac{\zeta_{r-1}^{nj}}{((\frac{p-1}{N_2})!)^k} (\zeta_p - 1)^{\frac{k(p-1)}{N_2}} \pmod{\tilde{\wp}^{\frac{k(p-1)}{N_2}+1}}. \end{aligned} \quad (4)$$

Since $\zeta_{r-1}^{nj} \pmod{\tilde{\wp}}$ ($j = 1, 2, \dots, N_2 - 1$) are all different, $\sum_{\chi^{N_2=1}, \chi \neq 1} g(\chi) \tilde{\chi}(\zeta_{r-1}^j \pmod{\wp})$ ($j = 1, 2, \dots, N_2 - 1$) are all different. Then the number of distinct nonzero $wt(c_2(\beta))$ is N_2 . This theorem is proved. ■

From Theorem 3.8, we can get a divisible property of a nonzero codeword in $\mathcal{C}(r, N)$.

Theorem 3.9: Let $p \equiv 1 \pmod{N_2}$ and $\beta \in GF(r)^*$, then $q^{\frac{N_2}{2}-1} || wt(c(\beta))$.

Proof: Note that p in $Z[\zeta_p]$ has the factorization

$$(p) = (\zeta_p - 1)^{p-1}.$$

Then this theorem can be obtained obviously from (4) in the proof of Theorem 3.8. ■

IV. THE WEIGHT DISTRIBUTIONS IN CASES $N_2 = 1, 2, 3, 4$

When $N_2 = 1$ or $N_2 = 2$, Ding [5] determined the weight distributions of $\mathcal{C}(r, N)$. His results are stated in Theorem 4.1 and Theorem 4.2.

Theorem 4.1: Let $N_2 = 1$, then $\mathcal{C}(r, N)$ is a $[(r-1)/N, m, q^{m-1}(q-1)/N]$ code with the only nonzero weight $q^{m-1}(q-1)/N$.

Theorem 4.2: Let $N_2 = 2$, then $\mathcal{C}(r, N)$ is a $[(r-1)/N, m, (q-1)(r-r^{1/2})/Nq]$ two weight code with the weight distribution

$$1 + \frac{r-1}{2} x^{(q-1)(r-r^{1/2})/Nq} + \frac{r-1}{2} x^{(q-1)(r+r^{1/2})/Nq}.$$

Theorem 4.3: Let $N_2 = 3$ and $p \equiv 1 \pmod{3}$, then $k \equiv 0 \pmod{3}$ and $\mathcal{C}(r, N)$ is a $[(r-1)/N, m]$ code with the weight distribution

$$\begin{aligned} 1 + \frac{r-1}{3} x^{(q-1)(r-cr^{1/3})/Nq} + \frac{r-1}{3} x^{(q-1)(r+\frac{1}{2}(c+9d)r^{1/3})/Nq} \\ + \frac{r-1}{3} x^{(q-1)(r+\frac{1}{2}(c-9d)r^{1/3})/Nq}, \end{aligned}$$

where c and d satisfy $4p^{k/3} = c^2 + 27d^2$, $c \equiv 1 \pmod{3}$ and $\gcd(c, p) = 1$.

Proof: From Theorem 3.5 and (a) in Lemma 2.2, this theorem follows immediately. ■

Theorem 4.4: Let $N_2 = 3$ and $p \equiv 2 \pmod{3}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, N)$ is a $[(r-1)/N, m]$ two weight code with the weight distribution

$$\begin{aligned} 1 + \frac{r-1}{3} x^{(q-1)(r+(-1)^{k/2}2r^{1/2})/Nq} \\ + \frac{2(r-1)}{3} x^{(q-1)(r-(-1)^{k/2}2r^{1/2})/Nq}. \end{aligned}$$

Proof: From Theorem 3.5 and (b) in Lemma 2.2, this theorem follows immediately. ■

Theorem 4.5: Let $N_2 = 4$ and $p \equiv 1 \pmod{4}$, then $k \equiv 0 \pmod{4}$ and $\mathcal{C}(r, N)$ is a $[(r-1)/N, m]$ code with the weight distribution

$$\begin{aligned} 1 + \frac{r-1}{4} x^{(q-1)(r+\sqrt{r}+2r^{1/4}u)/Nq} \\ + \frac{r-1}{4} x^{(q-1)(r+\sqrt{r}-2r^{1/4}u)/Nq} \\ + \frac{r-1}{4} x^{(q-1)(r-\sqrt{r}+4r^{1/4}v)/Nq} \\ + \frac{r-1}{4} x^{(q-1)(r-\sqrt{r}-4r^{1/4}v)/Nq}, \end{aligned}$$

where u and v satisfy $p^{k/2} = u^2 + 4v^2$, $u \equiv 1 \pmod{4}$ and $\gcd(u, p) = 1$.

Proof: From Theorem 3.5 and (a) in Lemma 2.3, this theorem follows immediately. ■

Theorem 4.6: Let $N_2 = 4$ and $p \equiv 3 \pmod{4}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, N)$ is a $[(r-1)/N, m]$ two weight code with the weight distribution

$$\begin{aligned} 1 + \frac{r-1}{4} x^{(q-1)(r+(-1)^{k/2}3r^{1/2})/Nq} \\ + \frac{3(r-1)}{4} x^{(q-1)(r-(-1)^{k/2}r^{1/2})/Nq}. \end{aligned}$$

Proof: From Theorem 3.5 and (b) in Lemma 2.3, this theorem follows immediately. ■

V. THE WEIGHT DISTRIBUTIONS IN CASES $N=5, 6, 8, 12$

Theorem 5.1: Let $N = 5$ and $N_2 = 1$, then $\mathcal{C}(r, 5)$ is a $[(r-1)/5, m, q^{m-1}(q-1)/5]$ code with the only nonzero weight $q^{m-1}(q-1)/5$.

Proof: From the weight distribution when $N_2 = 1$ in Ding [5], This theorem follows immediately. ■

Example Let $q = 11$ and let $m = 2$. Then the set $\mathcal{C}(r, 5)$ is a $[24, 2, 22]$ code over $GF(11)$ with the weight distribution

$$1 + 120x^{22}$$

Theorem 5.2: Let $N = 5$, $N_2 = 5$ and $p \equiv 1 \pmod{5}$, then \mathcal{C} is a $[(r-1)/5, m]$ code with weight list

$$\begin{aligned} wl = \\ [0, \frac{q-1}{qN} (r + \frac{\sqrt[5]{r}}{16} (x^3 - 25L)), \frac{q-1}{qN} (r - \frac{\sqrt[5]{r}}{64} (x^3 - 25M)), \\ \frac{q-1}{qN} (r - \frac{\sqrt[5]{r}}{64} \sigma (x^3 - 25M)), \frac{q-1}{qN} (r - \frac{\sqrt[5]{r}}{64} \sigma^2 (x^3 - 25M)), \\ \frac{q-1}{qN} (r - \frac{\sqrt[5]{r}}{64} \sigma^3 (x^3 - 25M))] \end{aligned}$$

and the weight distribution frequency

$$Freq(0) = 1, Freq(wl[i]) = \frac{r-1}{5} \quad (i = 2, \dots, 6).$$

Proof: From Theorem 3.5 and (a) in Lemma 2.4, this theorem follows immediately. ■

Example Let $q = 11$ and let $m = 5$. Then the set $\mathcal{C}(r, 5)$ is a $[32210, 5, 29050]$ code over $GF(11)$ with the weight distribution

$$1 + 32210x^{29050} + 32210x^{29200} + 32210x^{29300} \\ + 32210x^{29400} + 32210x^{29460}$$

Theorem 5.3: Let $N = 5$, $N_2 = 5$ and $p \not\equiv 1 \pmod{5}$. Let $k = sm$.

(a) If $k/\text{ord}_5 p$ is odd, $\mathcal{C}(r, 5)$ is a $[(r-1)/5, m, (q-1)(r-4r^{1/2})/5q]$ code with the weight distribution

$$1 + \frac{r-1}{5}x^{(q-1)(r-4r^{1/2})/5q} + \frac{4(r-1)}{5}x^{(q-1)(r+r^{1/2})/5q}.$$

(b) If $k/\text{ord}_5 p$ is even, $\mathcal{C}(r, 5)$ is a $[(r-1)/5, m, (q-1)(r-r^{1/2})/5q]$ code with the weight distribution

$$1 + \frac{4(r-1)}{5}x^{(q-1)(r-r^{1/2})/5q} + \frac{r-1}{5}x^{(q-1)(r+4r^{1/2})/5q}.$$

Proof: This theorem is the semi-primitive case in [2]. ■

Example Let $q = 7^2$ and let $m = 2$. Then the set $\mathcal{C}(r, 5)$ is a $[480, 2, 432]$ code over $GF(7^2)$ with the weight distribution

$$1 + 480x^{432} + 1920x^{480}$$

Let $q = 19$ and let $m = 4$. Then the set $\mathcal{C}(r, 5)$ is a $[26064, 4, 24624]$ code over $GF(19)$ with the weight distribution

$$1 + 104256x^{24624} + 26064x^{24966}$$

Theorem 5.4: Let $N = 6$ and $N_2 = 1$, then $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m, q^{m-1}(q-1)/6]$ code with the only nonzero weight $q^{m-1}(q-1)/6$.

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 1$. ■

Example Let $q = 7$ and let $m = 5$. Then the set $\mathcal{C}(r, 6)$ is a $[2801, 5, 2401]$ code over $GF(7)$ with the weight distribution

$$1 + 16806x^{2401}$$

Theorem 5.5: Let $N = 6$ and $N_2 = 2$, then $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m, (q-1)(r-r^{1/2})/6q]$ two weight code with the weight distribution

$$1 + \frac{r-1}{2}x^{(q-1)(r-r^{1/2})/6q} + \frac{r-1}{2}x^{(q-1)(r+r^{1/2})/6q}.$$

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 2$. ■

Example Let $q = 7$ and let $m = 2$. Then the set $\mathcal{C}(r, 6)$ is a $[8, 2, 6]$ code over $GF(7)$ with the weight distribution

$$1 + 24x^6 + 24x^8$$

Theorem 5.6: Let $N = 6$, $N_2 = 3$ and $p \equiv 1 \pmod{3}$, then $k \equiv 0 \pmod{3}$ and $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m]$ code with the weight distribution

$$1 + \frac{r-1}{3}x^{(q-1)(r-cr^{1/3})/6q} + \frac{r-1}{3}x^{(q-1)(r+\frac{1}{2}(c+9d)r^{1/3})/6q} \\ + \frac{r-1}{3}x^{(q-1)(r+\frac{1}{2}(c-9d)r^{1/3})/6q},$$

where c and d satisfy $4p^{k/3} = c^2 + 27d^2$, $c \equiv 1 \pmod{3}$ and $\gcd(c, p) = 1$.

Proof: This theorem follows immediately from the proof in Theorem 4.3. ■

Example Let $q = 7$ and let $m = 3$. Then the set $\mathcal{C}(r, 6)$ is a $[57, 3, 45]$ code over $GF(7)$ with the weight distribution

$$1 + 114x^{45} + 114x^{48} + 114x^{54}$$

Theorem 5.7: Let $N = 6$, $N_2 = 3$ and $p \equiv 2 \pmod{3}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m]$ two weight code with the weight distribution

$$1 + \frac{r-1}{3}x^{(q-1)(r+(-1)^{k/2}2r^{1/2})/6q} \\ + \frac{2(r-1)}{3}x^{(q-1)(r-(-1)^{k/2}2r^{1/2})/6q}.$$

Proof: This theorem follows immediately from the proof in Theorem 4.4. ■

Example Let $q = 5^2$ and let $m = 3$. Then the set $\mathcal{C}(r, 6)$ is a $[2604, 3, 2460]$ code over $GF(5^2)$ with the weight distribution

$$1 + 5208x^{2460} + 10416x^{2520}$$

Theorem 5.8: Let $N = 6$, $N_2 = 6$ and $p \equiv 1 \pmod{6}$, then $k \equiv 0 \pmod{6}$ and $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m]$ code with the weight distribution

$$1 + \frac{r-1}{6}x^{(q-1)(r-\eta_1^{*(6,r)})/6q} + \dots + \frac{r-1}{6}x^{(q-1)(r-\eta_6^{*(6,r)})/6q}.$$

Proof: From Theorem 3.5 and (a) in Lemma 2.5, this theorem follows immediately. ■

Example Let $q = 7$ and let $m = 6$. Then the set $\mathcal{C}(r, 6)$ is a $[19608, 6, 16596]$ code over $GF(7)$ with the weight distribution

$$1 + 19608x^{16596} + 19608x^{16776} + 19608x^{16812} \\ + 19608x^{16836} + 19608x^{16866} + 19608x^{16956}$$

Theorem 5.9: Let $N = 6$, $N_2 = 6$ and $p \equiv 5 \pmod{6}$, then $\mathcal{C}(r, 6)$ is a $[(r-1)/6, m]$ two weight code with the weight distribution

$$1 + \frac{5(r-1)}{6}x^{(q-1)(r-(-1)^{k/2}p^{k/2})/6q} \\ + \frac{r-1}{6}x^{(q-1)(r+5(-1)^{k/2}p^{k/2})/6q}.$$

Proof: This theorem is the semi-primitive case in [2]. ■

Example Let $q = 11$ and let $m = 2$. Then the set $\mathcal{C}(r, 6)$ is a $[20, 2, 10]$ code over $GF(11)$ with the weight distribution

$$1 + 20x^{10} + 100x^{20}$$

Theorem 5.10: Let $N = 8$ and $N_2 = 1$, then $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m, q^{m-1}(q-1)/8]$ code with the only nonzero weight $q^{m-1}(q-1)/8$.

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 1$. ■

Example Let $q = 17$ and let $m = 3$. Then the set $\mathcal{C}(r, 8)$ is a $[614, 3, 578]$ code over $GF(17)$ with the weight distribution

$$1 + 4912x^{578}$$

Theorem 5.11: Let $N = 8$ and $N_2 = 2$, then $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m, (q-1)(r-r^{1/2})/8q]$ two weight code with the weight distribution

$$1 + \frac{r-1}{2}x^{(q-1)(r-r^{1/2})/8q} + \frac{r-1}{2}x^{(q-1)(r+r^{1/2})/8q}.$$

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 2$. ■

Example Let $q = 7^2$ and let $m = 2$. Then the set $\mathcal{C}(r, 8)$ is a $[300, 2, 288]$ code over $GF(7^2)$ with the weight distribution

$$1 + 1200x^{288} + 1200x^{300}$$

Theorem 5.12: Let $N = 8$, $N_2 = 4$ and $p \equiv 1 \pmod{4}$, then $k \equiv 0 \pmod{4}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ code with the weight distribution

$$\begin{aligned} &1 + \frac{r-1}{4}x^{(q-1)(r+\sqrt{r}+2r^{1/4}u)/8q} \\ &+ \frac{r-1}{4}x^{(q-1)(r+\sqrt{r}-2r^{1/4}u)/8q} \\ &+ \frac{r-1}{4}x^{(q-1)(r-\sqrt{r}+4r^{1/4}v)/8q} \\ &+ \frac{r-1}{4}x^{(q-1)(r-\sqrt{r}-4r^{1/4}v)/8q}, \end{aligned}$$

where u and v satisfy $p^{k/2} = u^2 + 4v^2$, $u \equiv 1 \pmod{4}$ and $\gcd(u, p) = 1$.

Proof: This theorem follows immediately from the proof in Theorem 4.5. ■

Example Let $q = 17$ and let $m = 4$. Then the set $\mathcal{C}(r, 8)$ is a $[10440, 4, 9760]$ code over $GF(17)$ with the weight distribution

$$1 + 20880x^{9760} + 20880x^{9800} + 20880x^{9824} + 20880x^{9920}$$

Theorem 5.13: Let $N = 8$, $N_2 = 4$ and $p \equiv 3 \pmod{4}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ two weight code with the weight distribution

$$\begin{aligned} &1 + \frac{r-1}{4}x^{(q-1)(r+(-1)^{k/2}3r^{1/2})/8q} \\ &+ \frac{3(r-1)}{4}x^{(q-1)(r-(-1)^{k/2}r^{1/2})/8q}. \end{aligned}$$

Proof: This theorem follows immediately from the proof in Theorem 4.6. ■

Example Let $q = 11$ and let $m = 2$. Then the set $\mathcal{C}(r, 8)$ is a $[15, 2, 10]$ code over $GF(11)$ with the weight distribution

$$1 + 30x^{10} + 90x^{15}$$

Theorem 5.14: Let $N = 8$, $N_2 = 8$ and $p \equiv 1 \pmod{8}$, then $k \equiv 0 \pmod{8}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ code with the weight distribution

$$1 + \frac{r-1}{8}x^{(q-1)(r-\eta_1^{*(s,r)})/8q} + \dots + \frac{r-1}{8}x^{(q-1)(r-\eta_8^{*(s,r)})/8q},$$

where $\eta_j^{*(s,r)} = -p^{k/2} - p^{k/4}Q_{j,k/2} - p^{k/8}AB$ and A, B are defined as follows: If j is even, $A = Q_{j/2,k/4}$ and $B = T_{k/2}$; if j is odd, $A = (-1)^{[j/4]}Q_{0,k/4} + (-1)^{[(j-2)/4]}P_{0,k/4}$ and $B = S_{k/2}$.

Proof: From Theorem 3.5 and (a) in Lemma 2.6, this theorem follows immediately. ■

Example Let $q = 17$ and let $m = 8$. Then the set $\mathcal{C}(r, 8)$ is a $[871969680, 8, 820657856]$ code over $GF(17)$ with the weight list

$$wl := [0, 820657856, 820663680, 820666436, 820675268, 820694592, 820702148, 820704836, 820732560]$$

and the weight distribution frequency

$$Freq(0) = 1$$

$$Freq(wl[i]) = 871969680, i = 2, \dots, 9.$$

Theorem 5.15: Let $N = 8$, $N_2 = 8$ and $p \equiv 3 \pmod{8}$, then $k \equiv 0 \pmod{4}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ code with the weight distribution

$$\begin{aligned} &1 + \frac{r-1}{4}x^{(q-1)(r-\xi_1)/8q} + \frac{r-1}{4}x^{(q-1)(r-\xi_2)/8q} \\ &+ \frac{r-1}{4}x^{(q-1)(r-\xi_3)/8q} + \frac{r-1}{8}x^{(q-1)(r-\xi_4)/8q} \\ &+ \frac{r-1}{8}x^{(q-1)(r-\xi_5)/8q}, \end{aligned}$$

where $\xi_1 = -2p^{k/2}S_{k/2} + p^{k/2}$, $\xi_2 = 2p^{k/2}S_{k/2} + p^{k/2}$, $\xi_3 = p^{k/2}$, $\xi_4 = 2p^{k/4}T_{k/2} - 3p^{k/2}$ and $\xi_5 = -2p^{k/4}T_{k/2} - 3p^{k/2}$.

Proof: From Theorem 3.5 and (b) in Lemma 2.6, this theorem follows immediately. ■

Example Let $q = 3$ and let $m = 4$. Then the set $\mathcal{C}(r, 8)$ is a $[10, 4, 4]$ code over $GF(3)$ with the weight distribution

$$1 + 20x^4 + 20x^6 + 30x^8 + 10x^{10}$$

Theorem 5.16: Let $N = 8$, $N_2 = 8$ and $p \equiv 5 \pmod{8}$, then $k \equiv 0 \pmod{8}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ code with the weight distribution

$$\begin{aligned} &1 + \frac{r-1}{4}x^{(q-1)(r-\xi_1)/8q} + \frac{r-1}{4}x^{(q-1)(r-\xi_2)/8q} \\ &+ \frac{r-1}{8}x^{(q-1)(r-\xi_3)/8q} + \frac{r-1}{8}x^{(q-1)(r-\xi_4)/8q} \\ &+ \frac{r-1}{8}x^{(q-1)(r-\xi_5)/8q} + \frac{r-1}{8}x^{(q-1)(r-\xi_6)/8q}, \end{aligned}$$

where $\xi_1 = p^{k/2} - p^{k/4}P_{k/2}$, $\xi_2 = p^{k/2} + p^{k/4}P_{k/2}$, $\xi_3 = -p^{k/2} - p^{k/4}Q_{k/2} - 2p^{3k/8}Q_{k/4}$, $\xi_4 = -p^{k/2} - p^{k/4}Q_{k/2} + 2p^{3k/8}Q_{k/4}$, $\xi_5 = -p^{k/2} + p^{k/4}Q_{k/2} - 2p^{3k/8}P_{k/4}$ and $\xi_6 = -p^{k/2} + p^{k/4}Q_{k/2} + 2p^{3k/8}P_{k/4}$.

Proof: From Theorem 3.5 and (c) in Lemma 2.6, this theorem follows immediately. ■

Example Let $q = 5$ and let $m = 8$. Then the set $\mathcal{C}(r, 8)$ is a $[48828, 8, 38880]$ code over $GF(5)$ with the weight distribution

$$\begin{aligned} &1 + 97656x^{38880} + 48828x^{38940} + 48828x^{38960} \\ &+ 97656x^{39120} + 48828x^{39240} + 48828x^{39360} \end{aligned}$$

Theorem 5.17: Let $N = 8$, $N_2 = 8$ and $p \equiv 7 \pmod{8}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 8)$ is a $[(r-1)/8, m]$ two weight code with the weight distribution

$$1 + \frac{7(r-1)}{8}x^{(q-1)(r-(-1)^{k/2}p^{k/2})/8q} + \frac{r-1}{8}x^{(q-1)(r+7(-1)^{k/2}p^{k/2})/8q}.$$

Proof: This theorem is the semi-primitive case in [2]. ■

Example Let $q = 7$ and let $m = 4$. Then the set $\mathcal{C}(r, 8)$ is a $[300, 4, 252]$ code over $GF(7)$ with the weight distribution

$$1 + 2100x^{252} + 300x^{294}$$

Theorem 5.18: Let $N = 12$ and $N_2 = 1$, then $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m, q^{m-1}(q-1)/12]$ code with the only nonzero weight $q^{m-1}(q-1)/12$.

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 1$. ■

Example Let $q = 13$ and let $m = 5$. Then the set $\mathcal{C}(r, 12)$ is a $[30941, 5, 28561]$ code over $GF(13)$ with the weight distribution

$$1 + 371292x^{28561}$$

Theorem 5.19: Let $N = 12$ and $N_2 = 2$, then $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m, (q-1)(r-r^{1/2})/12q]$ two weight code with the weight distribution

$$1 + \frac{r-1}{2}x^{(q-1)(r-r^{1/2})/12q} + \frac{r-1}{2}x^{(q-1)(r+r^{1/2})/12q}.$$

Proof: This theorem follows immediately from Ding [5] for the case $N_2 = 2$. ■

Example Let $q = 5^2$ and let $m = 2$. Then the set $\mathcal{C}(r, 12)$ is a $[52, 2, 48]$ code over $GF(5^2)$ with the weight distribution

$$1 + 312x^{48} + 312x^{52}$$

Theorem 5.20: Let $N = 12$, $N_2 = 3$ and $p \equiv 1 \pmod{3}$, then $k \equiv 0 \pmod{3}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{3}x^{(q-1)(r-cr^{1/3})/12q} + \frac{r-1}{3}x^{(q-1)(r+\frac{1}{2}(c+9d)r^{1/3})/12q} + \frac{r-1}{3}x^{(q-1)(r+\frac{1}{2}(c-9d)r^{1/3})/12q},$$

where c and d satisfy $4p^{k/3} = c^2 + 27d^2$, $u \equiv 1 \pmod{3}$ and $\gcd(c, p) = 1$.

Proof: This theorem follows immediately from Theorem 4.3. ■

Example Let $q = 13$ and let $m = 3$. Then the set $\mathcal{C}(r, 12)$ is a $[183, 3, 162]$ code over $GF(13)$ with the weight distribution

$$1 + 732x^{162} + 732x^{171} + 732x^{174}$$

Theorem 5.21: Let $N = 12$, $N_2 = 3$ and $p \equiv 2 \pmod{3}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ two weight code with the weight distribution

$$1 + \frac{r-1}{3}x^{(q-1)(r+(-1)^{k/2}2r^{1/2})/12q} + \frac{2(r-1)}{3}x^{(q-1)(r-(-1)^{k/2}r^{1/2})/12q}.$$

Proof: This theorem follows immediately from Theorem 4.4. ■

Example Let $q = 5^2$ and let $m = 3$. Then the set $\mathcal{C}(r, 12)$ is a $[1302, 3, 1230]$ code over $GF(5^2)$ with the weight distribution

$$1 + 5202x^{1230} + 10416x^{1260}$$

Theorem 5.22: Let $N = 12$, $N_2 = 4$ and $p \equiv 1 \pmod{4}$, then $k \equiv 0 \pmod{4}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{4}x^{(q-1)(r+\sqrt{r}+2r^{1/4}u)/12q} + \frac{r-1}{4}x^{(q-1)(r+\sqrt{r}-2r^{1/4}u)/12q} + \frac{r-1}{4}x^{(q-1)(r-\sqrt{r}+4r^{1/4}v)/12q} + \frac{r-1}{4}x^{(q-1)(r-\sqrt{r}-4r^{1/4}v)/12q},$$

where u and v satisfy $p^{k/2} = u^2 + 4v^2$, $u \equiv 1 \pmod{4}$ and $\gcd(u, p) = 1$.

Proof: This theorem follows immediately from Theorem 4.5. ■

Example Let $q = 13$ and let $m = 4$. Then the set $\mathcal{C}(r, 12)$ is a $[2380, 4, 2160]$ code over $GF(13)$ with the weight distribution

$$1 + 7140x^{2160} + 7140x^{2200} + 7140x^{2208} + 7140x^{2220}$$

Theorem 5.23: Let $N = 12$, $N_2 = 4$ and $p \equiv 3 \pmod{4}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ two weight code with the weight distribution

$$1 + \frac{r-1}{4}x^{(q-1)(r+(-1)^{k/2}3r^{1/2})/12q} + \frac{3(r-1)}{4}x^{(q-1)(r-(-1)^{k/2}r^{1/2})/12q}.$$

Proof: This theorem follows immediately from Theorem 4.6. ■

Example Let $q = 7$ and let $m = 2$. Then the set $\mathcal{C}(r, 12)$ is a $[4, 2, 2]$ code over $GF(7)$ with the weight distribution

$$1 + 12x^2 + 36x^4$$

Theorem 5.24: Let $N = 12$, $N_2 = 6$ and $p \equiv 1 \pmod{6}$, then $k \equiv 0 \pmod{6}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{6}x^{(q-1)(r-\eta_1^{*(6,r)})/12q} + \dots + \frac{r-1}{6}x^{(q-1)(r-\eta_6^{*(6,r)})/12q}.$$

Proof: From Theorem 3.5 and (a) in Lemma 2.5, this theorem follows immediately. ■

Example Let $q = 13$ and let $m = 6$. Then the set $\mathcal{C}(r, 12)$ is a $[402234, 6, 370692]$ code over $GF(13)$ with the weight list

$$wl := [0, 370692, 371112, 371232, 371322, 371448, 371952]$$

and the weight distribution frequency

$$Freq(0) = 1$$

$$Freq(wl[i]) = 804468, i = 2, \dots, 7.$$

Theorem 5.25: Let $N = 12$, $N_2 = 6$ and $p \equiv 5 \pmod{6}$, then $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ two weight code with the weight distribution

$$1 + \frac{5(r-1)}{6}x^{(q-1)(r-(-1)^{k/2}p^{k/2})/12q} + \frac{r-1}{6}x^{(q-1)(r+5(-1)^{k/2}p^{k/2})/12q}.$$

Proof: This theorem is the semi-primitive case in [2]. ■

Example Let $q = 17$ and let $m = 2$. Then the set $\mathcal{C}(r, 12)$ is a $[24, 2, 16]$ code over $GF(17)$ with the weight distribution

$$1 + 48x^{16} + 240x^{24}$$

Theorem 5.26: Let $N = 12$, $N_2 = 12$ and $p \equiv 1 \pmod{12}$, then $k \equiv 0 \pmod{12}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{12}x^{(q-1)(r-\eta_1^{*(12,r)})/12q} + \dots + \frac{r-1}{12}x^{(q-1)(r-\eta_{12}^{*(12,r)})/12q},$$

where $\eta_j^{*(12,r)} = -p^{k/12}Q_{j,k/2}V_{-j,k/3} - p^{k/4}Q_{j,k/2} - p^{k/6}V_{j,2k/3} - p^{k/3}V_{2j,k/3} - (-1)^j p^{k/2}$.

Proof: From Theorem 3.5 and (a) in Lemma 2.7, this theorem follows immediately. ■

Example Let $q = 13$ and let $m = 12$. Then the set $\mathcal{C}(r, 12)$ is a $[1941507093540, 12, 1792157710608]$ code over $GF(13)$ with the weight list

$$\begin{aligned} wl := & [0, 1792157710608, 1792159338564, 1792159386480, \\ & 1792159451424, 1792160074992, 1792160674272, \\ & 1792160747136, 1792160770896, 1792160847072, \\ & 1792161442512, 1792161902664, 1792162381824] \end{aligned}$$

and the weight distribution frequency

$$\begin{aligned} Freq(0) &= 1 \\ Freq(wl[i]) &= 1941507093540, i = 2, \dots, 13. \end{aligned}$$

Theorem 5.27: Let $N = 12$, $N_2 = 12$ and $p \equiv 5 \pmod{12}$, then $k \equiv 0 \pmod{4}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{6}x^{(q-1)(r-\xi_1)/12q} + \frac{r-1}{6}x^{(q-1)(r-\xi_2)/12q} + \frac{r-1}{6}x^{(q-1)(r-\xi_3)/12q} + \frac{r-1}{6}x^{(q-1)(r-\xi_4)/12q} + \frac{r-1}{12}x^{(q-1)(r-\xi_5)/12q} + \dots + \frac{r-1}{12}x^{(q-1)(r-\xi_8)/12q},$$

where $\xi_1 = Q_{k/2}p^{k/4}((-1)^{k/4} - 1) + p^{k/2}$, $\xi_2 = -Q_{k/2}p^{k/4}((-1)^{k/4} - 1) + p^{k/2}$, $\xi_3 = P_{k/2}p^{k/4}((-1)^{k/4} + 1) + p^{k/2}$, $\xi_4 = -P_{k/2}p^{k/4}((-1)^{k/4} + 1) + p^{k/2}$, $\xi_5 = P_{k/2}p^{k/4}(2(-1)^{k/4} - 1) + p^{k/2}$, $\xi_6 = -P_{k/2}p^{k/4}(2(-1)^{k/4} - 1) + p^{k/2}$, $\xi_7 = Q_{k/2}p^{k/4}(2(-1)^{k/4} + 1) - 5p^{k/2}$ and $\xi_8 = -Q_{k/2}p^{k/4}(2(-1)^{k/4} - 1) - 5p^{k/2}$.

Proof: From Theorem 3.5 and (b) in Lemma 2.7, this theorem follows immediately. ■

Example Let $q = 5$ and let $m = 4$. Then the set $\mathcal{C}(r, 12)$ is a $[52, 4, 32]$ code over $GF(5)$ with the weight distribution

$$1 + 52x^{32} + 104x^{36} + 208x^{40} + 104x^{44} + 104x^{48} + 52x^{52}.$$

Theorem 5.28: Let $N = 12$, $N_2 = 12$ and $p \equiv 7 \pmod{12}$. Let $\rho = 2(-1)^{k(p+5)/6}$, then $k \equiv 0 \pmod{6}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ code with the weight distribution

$$1 + \frac{r-1}{12}x^{(q-1)(r-\eta_1^{*(12,r)})/12q} + \dots + \frac{r-1}{12}x^{(q-1)(r-\eta_{12}^{*(12,r)})/12q},$$

where $\eta_j^{*(12,r)}$ is defined as follows: If j is odd, $\eta_j^{*(12,r)} = -(-1)^{k/2}p^{k/6}V_{j,2k/3} - p^{k/3}V_{2j,k/3} + (-1)^{k/2}p^{k/2}$; if $2||j$, $\eta_j^{*(12,r)} = -(-1)^{k/2}p^{k/6}V_{j,2k/3} + p^{k/3}V_{2j,k/3}(\rho - 1) + p^{k/2}(\rho - (-1)^{k/2})$; if $4|j$, $\eta_j^{*(12,r)} = -(-1)^{k/2}p^{k/6}V_{j,2k/3} - p^{k/3}(\rho + 1)V_{2j,k/3} - p^{k/2}(\rho + (-1)^{k/2})$.

Proof: From Theorem 3.5 and (c) in Lemma 2.7, this theorem follows immediately. ■

Example Let $q = 7$ and let $m = 6$. Then the set $\mathcal{C}(r, 12)$ is a $[9804, 6, 8256]$ code over $GF(7)$ with the weight distribution

$$\begin{aligned} & 1 + 9804x^{8256} + 9804x^{8280} + 9804x^{8340} \\ & + 9804x^{8730} + 19608x^{8388} + 19608x^{8418} \\ & + 19608x^{8478} + 9804x^{8496} + 9804x^{8532}. \end{aligned}$$

Theorem 5.29: Let $N = 12$, $N_2 = 12$ and $p \equiv 11 \pmod{12}$, then $k \equiv 0 \pmod{2}$ and $\mathcal{C}(r, 12)$ is a $[(r-1)/12, m]$ two weight code with the weight distribution

$$1 + \frac{11(r-1)}{12}x^{(q-1)(r-(-1)^{k/2}p^{k/2})/12q} + \frac{r-1}{12}x^{(q-1)(r+11(-1)^{k/2}p^{k/2})/12q}.$$

Proof: This theorem is the semi-primitive case in [2]. ■

Example Let $q = 23$ and let $m = 2$. Then the set $\mathcal{C}(r, 12)$ is a $[44, 2, 22]$ code over $GF(23)$ with the weight distribution

$$1 + 44x^{22} + 484x^{44}$$

VI. CONCLUSION

This paper presents necessary and sufficient conditions for an irreducible cyclic code having only one nonzero weight or the maximal number of distinct nonzero weights. When the number of distinct nonzero weight achieves the maximum, we obtain a divisible property of a codeword. Further, we determine the weight distributions of $\mathcal{C}(r, N)$ for cases $N_2 = 3, 4$ or $N = 5, 6, 8, 12$. From Theorem 3.5, we can immediately obtain the weight distribution of $\mathcal{C}(r, N)$ for cases $N_2 = 5, 6, 8, 12$.

ACKNOWLEDGMENT

This paper is supported by National Natural Science Foundation of China (61003285), the Fundamental Research Funds for the Central Universities (BUPT2009RC0215) and the Fundamental Research Funds for the Central Universities (BUPT2011RC0209).

REFERENCES

- [1] Y. Aubry and P. Langevin, On the weights of binary irreducible cyclic codes, in Proc. Workshop on Coding and Cryptography, Bergen, Norway, 2005, pp. 161-169.
- [2] L. D. Baumert and R. J. McEliece, Weights of irreducible cyclic codes, Inf. Contr., vol. 20, no. 2, pp. 158-175, 1972.
- [3] L. D. Baumert and J. Mykkeltveit, Weight distributions of some irreducible cyclic codes, DSN Progr. Rep., vol. 16, pp. 128-131, 1973.
- [4] P. Delsarte and J. M. Goethals, Irreducible binary cyclic codes of even dimension, in Proc. 2nd Chapel Hill Conf. Combinatorial Mathematics and Its Applications, Chapel Hill, NC, 1970, pp. 100-113.
- [5] C. Ding, the Weight Distribution of Some Irreducible Cyclic Codes, IEEE Transactions on Information Theory, Vol 55, NO. 3, March 2009.
- [6] C. Ding, Niederreiter H., Cyclotomic linear codes of order 3. IEEE Trans. Inform. Theory 53, 2274-2277(2007).
- [7] S. Gurak, Period polynomials for F_q of fixed small degree, in Number Theory, Providence, RI: Amer. Math. Soc., 2004, pp. 127-145.
- [8] T. Hellese, T. Klove, and J. Mykkeltveit, The weight distribution of irreducible cyclic codes with block length $n_1((q^l-1)/N)$, Discr. Math., vol. 18, no. 2, pp. 179-211, 1977.
- [9] A. Hoshi, Explicit lifts of quintic Jacobi sums and period polynomials for F_q , Proc. Japan Acad., vol. 82, no. 7, pp. 87-92, 2006.
- [10] S. Lang, Cyclotomic Fields, Graduate Texts in Mathematics, Springer-Verlag: New York, 1978.
- [11] S. Lang, Cyclotomic Fields, II. Graduate Texts in Mathematics, Springer-Verlag: New York, 1980.
- [12] P. Langevin, A new class of two weight codes, in Finite Fields and Applications, S. Cohen and H. Niederreiter, Eds. Cambridge, U.K.: Cambridge Univ. Press, 1996, pp. 181-187.
- [13] R. Lidl and H. Niederreiter, Introduction to Finite Fields and Their applications, Cambridge University Press, Cambridge, UK, 1994.
- [14] F. MacWilliams and J. Seery, The weight distributions of some minimal cyclic codes, IEEE Trans. Inf. Theory, vol. IT-27, no. 6, pp. 796-806, Nov. 1981.
- [15] R. J. McEliece, A class of two-weight codes, Jet Propulsion Laboratory Space Program Summary 37C41, vol. IV, pp. 264-266.
- [16] R. J. McEliece, Irreducible cyclic codes and Gauss sums, in Combinatorics, Part 1: Theory of Designs, Finite Geometry and Coding Theory. Amsterdam, The Netherlands: Math. Centrum, 1974, vol. 55, Math. Centre Tracts, pp. 179-196.
- [17] R. J. McEliece and H. Rumsey Jr., Euler products, cyclotomy, and coding, J. Number Theory, vol. 4, pp. 302-311, 1972.
- [18] M. J. Moisio and K. O. Vaananen, Two recursive algorithms for computing the weight distribution of certain irreducible cyclic codes, IEEE Trans. Inf. Theory, vol. 45, no. 3, pp. 1244-1249, May 1999.
- [19] G. Myerson, Period polynomials and Gauss sums for finite fields, Acta Arith., vol. 39, pp. 251-264, 1981.
- [20] B. Schmidt and C. White, All two-weight irreducible cyclic codes, Finite Fields Appl., vol. 8, pp. 1-17, 2002.
- [21] L. Stickelberger, Über eine Verallgemeinerung der Kreistheilung, Math. Annal. 37 (1890), 321-367.
- [22] M. van der Vlugt, On the weight hierarchy of irreducible cyclic codes, J. Comb. Theory Ser. A, vol. 71, no. 1, pp. 159-167, Jul. 1995.
- [23] L. C. Washington, Introduction to Cyclotomic Fields, Graduate Texts in Math., Vol. 83. Springer-Verlag, Berlin/New York/Heidelberg, 1997.